

The Landau-Lifshitz equation of the ferromagnetic spin chain and Oseen-Frank flow

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Abstract

In this paper, we consider the Landau-Lifshitz equation of the ferromagnetic spin chain from \mathbb{R}^2 to the unit sphere S^2 under the general Oseen-Frank energy. We obtain global existence and uniqueness of weak solutions for large energy data; moreover, the number of singular points is finite.

1 Introduction

The d -dimensional classical system for the isotropic Heisenberg chain with spin vector $\mathbf{n} = (n_1, n_2, n_3)$ is described by the Hamiltonian density (without external magnetic field) $H = |\nabla \mathbf{n}|^2/2$. The spin equation of motion with the Gilbert damping term (without the external magnetic field) has the form

$$\partial_t \mathbf{n} = \alpha \mathbf{n} \times (\mathbf{n} \times \frac{\delta H}{\delta \mathbf{n}}) - \beta \mathbf{n} \times \frac{\delta H}{\delta \mathbf{n}}, \quad (1.1)$$

where $\alpha \geq 0$ is the Gilbert damping constant and β is the exchange constant satisfying $\alpha^2 + \beta^2 = 1$ and H is the Hamiltonian density. Explicitly, this gives the following classical Landau-Lifshitz equation

$$\partial_t \mathbf{n} = \beta \mathbf{n} \times \Delta \mathbf{n} - \alpha \mathbf{n} \times (\mathbf{n} \times \Delta \mathbf{n}). \quad (1.2)$$

The above system (1.1) or (1.2) is called the Landau-Lifshitz equation or the Landau-Lifshitz-Gilbert equation, which was first derived on phenomenological grounds by Landau-Lifshitz in [22]. It gives rise to a continuum spin wave theory. Note that the above system (1.2) reduces to the heat flow of harmonic maps when $\alpha = 1, \beta = 0$ and to the Schrödinger flow when $\alpha = 0, \beta = 1$.

Motivated by the study on the heat flow of harmonic maps (see [17, 31, 32, 33, 9] and so on) and Schrödinger flow (see [13, 14, 34] and so on), much progress has been made recently in the analysis of the Landau-Lifshitz-Gilbert Equation (1.2). For example, see [2] for the existence of global weak solutions of (1.2) under the Neumann boundary condition in any dimensions, and see [10, 27, 12, 25, 28, 4, 15, 19] and the references therein for partial regularity and the analysis of singularity of the system (1.2). More recently, the existence of partially smooth, global weak solutions of (1.2) similar to [32], has been obtained by Guo-Hong [18] for $d = 2$, Melcher [26] for $d = 3$, and Wang [35] for $d = 4$ with Dirichlet

boundary conditions. More recently, the first author and Guo [29, 30] studied the fractional generalization of the Landau-Lifshitz equation and obtained local well-posedness and global existence of weak solutions.

In this paper, we shall consider the case when the energy density is replaced by the Oseen-Frank energy density. The Oseen-Frank energy density expresses the free energy density of a nematic liquid crystal in terms of its optic axis, and is a measure of the increase in the Helmholtz free energy per unit volume due to deviations in the orientational ordering away from a uniformly aligned nematic director configuration. See [20] for the analysis for the minimizers of the Oseen-Frank energy. Let $W = W(\mathbf{n}, \nabla \mathbf{n})$ be the Oseen-Frank density of the form

$$W(\mathbf{n}, \nabla \mathbf{n}) = k_1(\operatorname{div} \mathbf{n})^2 + k_2|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + k_3|\mathbf{n} \cdot (\nabla \times \mathbf{n})|^2 \\ + (k_2 + k_4)(\operatorname{tr}(\nabla \mathbf{n})^2 - (\operatorname{div} \mathbf{n})^2),$$

where k_1, k_2, k_3, k_4 are elastic constants depending on the materials and temperature.

Replacing H in (1.1) with W , we obtain the Landau-Lifshitz equation of Oseen-Frank energy as follows:

$$\partial_t \mathbf{n} = -\alpha \mathbf{n} \times (\mathbf{n} \times \mathbf{h}) + \beta \mathbf{n} \times \mathbf{h}, \quad (1.3)$$

where the vector field \mathbf{h} is given by

$$\mathbf{h} = -\frac{\delta W}{\delta \mathbf{n}} = (\nabla_i W_{p_i^l} - W_{n_l}),$$

where $p_i^l = \nabla_i n_l$ and we adopt the standard summation convention. Throughout this paper, we denote

$$W_{n_i} = \frac{\partial W(\mathbf{n}, \mathbf{p})}{\partial n_i}, \quad W_{p_i^l} = \frac{\partial W(\mathbf{n}, \mathbf{p})}{\partial p_i^l}.$$

In what follows, we give explicit form of the vector field \mathbf{h} . For this, we rewrite $W(\mathbf{n}, \nabla \mathbf{n})$ as in [20]

$$W(\mathbf{n}, \nabla \mathbf{n}) = a|\nabla \mathbf{n}|^2 + V(\mathbf{n}, \nabla \mathbf{n}),$$

where $a = \min\{k_1, k_2, k_3\}$ and

$$V(\mathbf{n}, \nabla \mathbf{n}) = (k_1 - a)(\operatorname{div} \mathbf{n})^2 + (k_2 - a)|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + (k_3 - a)|\mathbf{n} \cdot (\nabla \times \mathbf{n})|^2.$$

In this way, we have the following (see [36])

Lemma 1.1 *It holds that*

$$(\nabla_\alpha W_{p_\alpha^l}) = 2a\Delta \mathbf{n} + 2(k_1 - a)\nabla \operatorname{div} \mathbf{n} - 2(k_2 - a)\operatorname{curl}(\mathbf{n} \times (\operatorname{curl} \mathbf{n} \times \mathbf{n})) \\ - 2(k_3 - a)\operatorname{curl}(\operatorname{curl} \mathbf{n} \cdot \mathbf{n}), \\ (W_{n_l}) = 2(k_3 - k_2)(\operatorname{curl} \mathbf{n} \cdot \mathbf{n})(\operatorname{curl} \mathbf{n}),$$

In particular, we have

$$\mathbf{h} = 2a\Delta \mathbf{n} + 2(k_1 - a)\nabla \operatorname{div} \mathbf{n} - 2(k_2 - a)\operatorname{curl}(\operatorname{curl} \mathbf{n}) \\ - 2(k_3 - k_2)\operatorname{curl}(\operatorname{curl} \mathbf{n} \cdot \mathbf{n}) - 2(k_3 - k_2)(\operatorname{curl} \mathbf{n} \cdot \mathbf{n})(\operatorname{curl} \mathbf{n}). \quad (1.4)$$

In particular, when $k_1 = k_2 = k_3$, (1.3) with (1.4) reduces to the classical Landau-Lifshitz equation (1.2).

In [21], Hong-Xin proved that global existence of weak solution for the Oseen-Frank flow in 2D (i.e. $\alpha = 1, \beta = 0$ in (1.3)) whose singular points are finite and the uniqueness of weak solution was obtained by the later two authors of the present paper in [37] (see also [23] for different assumptions).

We are aimed to generalize the above results to the general Landau-Lifshitz equation (1.3) with $\alpha, \beta > 0$. Note that $\partial_{x_3} \mathbf{n} = 0$ in the 2-D case. Let $b \in S^2$ be a constant vector and we define

$$H_b^1(\mathbb{R}^2; S^2) = \{u : u - b \in H^1(\mathbb{R}^2; \mathbb{R}^3), |u| = 1 \text{ a.e. in } \mathbb{R}^2\}.$$

our main results state as follows.

Theorem 1.2 *Assume that the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2; S^2)$. Then there exists a unique global weak solution \mathbf{n} of the system (1.3), which is smooth in $\mathbb{R}^2 \times ((0, +\infty) \setminus \{T_i\}_{i=1}^L)$ with a finite number of singular points (x_i^l, T_i) , $1 \leq l \leq L_i$. Moreover, there are two constants $\epsilon_0 > 0$ and $R_0 > 0$ such that each singular point (x_i^l, T_i) is characterized by the condition*

$$\limsup_{t \uparrow T_i} \int_{B_R(x_i^l)} |\nabla \mathbf{n}|^2(\cdot, t) dx > \epsilon_0$$

for any $R > 0$ with $R \leq R_0$.

Remark 1.3 *The above theorem generalizes the existence and uniqueness results of the equation (1.2) in [18], and also generalize the existence result in [21]. The main difference is the introduced Oseen-Frank energy, which makes the system (1.3) does not keep the parabolic property. By constructing strong solutions of a new approximate system, we obtain the local well-posedness and global weak solutions of (1.3). Different with [32, 18], it's not easy to obtain the uniqueness as said in [21], since the positivity of the diffusion term $\delta_{\mathbf{h}} \times \mathbf{n}$ under the metric of L^2 norm is unknown. Instead, we introduce a type of weak Oseen-Frank metric as in [37]. Our goal is to combine the work of Oseen-Frank energy and the Schrödinger part $\mathbf{n} \times \mathbf{h}$ together.*

The rest of the paper is organized as follows. In Section 2, we obtain global existence of weak solution for the system (1.3) by using the local well-posedness and blow-up results in the Appendix. In Section 3, we prove that the weak solution obtained in Section 2 is unique indeed. At last, the local well-posedness and blow-up results for the Landau-Lifshitz system (1.3) with general Oseen-Frank energy are obtained in the Appendix.

2 Global existence of weak solutions in \mathbb{R}^2

Let $E(t) = \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) dx$ for $t \geq 0$ and $E_0 = E(0) = \int_{\mathbb{R}^2} W(\mathbf{n}_0, \nabla \mathbf{n}_0)(x) dx$. Moreover,

$$E_R(\mathbf{n}(\cdot, t); x) = \int_{B_R(x)} |\nabla \mathbf{n}(y, t)|^2 dy.$$

For two constants τ and T with $0 \leq \tau < T$, we denote

$$V(\tau, T) : = \{\mathbf{n} : \mathbb{R}^2 \times [\tau, T] \rightarrow S^2 \mid \mathbf{n} \text{ is measurable and satisfies} \\ \text{esssup}_{\tau \leq t \leq T} \int_{\mathbb{R}^2} |\nabla \mathbf{n}(\cdot, t)|^2 dx + \int_{\tau}^T \int_{\mathbb{R}^2} |\nabla^2 \mathbf{n}|^2 + |\partial_t \mathbf{n}|^2 dx dt < \infty\}.$$

2.1 A priori estimates

The following technical lemma can be found in [32].

Lemma 2.1 *There are constants C and R_0 such that for any $u \in V(0, T)$ and any $R \in (0, R_0]$, we have*

$$\begin{aligned} \int_{\mathbb{R}^2 \times [0, T]} |\nabla u|^4 dx dt &\leq C \text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla u(\cdot, t)|^2 dx \\ &\quad \cdot \left(\int_{\mathbb{R}^2 \times [0, T]} |\nabla^2 u|^2 + R^{-2} \int_{\mathbb{R}^2 \times [0, T]} |\nabla u|^2 dx dt \right). \end{aligned} \quad (2.1)$$

First of all, we have the following basic energy estimates.

Lemma 2.2 (The basic energy estimates) *Assume that \mathbf{n} is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2)$. Then, for all $0 < t < T$ there holds*

$$\int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) dx + \alpha \int_0^t \int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 dx ds \leq E_0.$$

Proof: Multiply $\partial_t \mathbf{n}$ on both sides of the equation (1.3) and integrate on \mathbb{R}^2 , then the property $|\mathbf{n}| = 1$ implies that

$$\begin{aligned} \int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 dx &= \alpha \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) dx + \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) dx \\ &= \alpha \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot \mathbf{h} dx + \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) dx \end{aligned}$$

Noting that the definition of the molecular field \mathbf{h} , we get

$$\frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) dx = \int_{\mathbb{R}^2} (-\mathbf{h}) \cdot \partial_t \mathbf{n} dx.$$

It follows that

$$\int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 dx + \alpha \frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) dx = \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) dx. \quad (2.2)$$

Now we estimate the term $\partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h})$ as in [18]. The equation (1.3) show that

$$\partial_t \mathbf{n} = \alpha \mathbf{n} \times (\mathbf{h} \times \mathbf{n}) + \beta \mathbf{n} \times \mathbf{h},$$

then we have

$$\mathbf{n} \times \partial_t \mathbf{n} = \alpha \mathbf{n} \times \mathbf{h} + \beta \mathbf{n} \times (\mathbf{n} \times \mathbf{h}),$$

hence using $\alpha^2 + \beta^2 = 1$ we arrive at

$$\mathbf{n} \times \partial_t \mathbf{n} + \frac{\beta}{\alpha} \partial_t \mathbf{n} = \frac{1}{\alpha} \mathbf{n} \times \mathbf{h},$$

which yields that

$$\partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) = \beta |\partial_t \mathbf{n}|^2. \quad (2.3)$$

Combining the estimates (2.2) and (2.3), we have

$$\alpha^2 \int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 dx + \alpha \frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) dx = 0,$$

and the proof is completed by integrating with respect to time. \square

As in [32, 18], the key ingredient for global existence of weak solution is a local monotonicity inequality, and our results state as follows.

Lemma 2.3 (The local monotonicity inequality) *Assume that \mathbf{n} is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2)$. Then, for all $0 < t < T$ and $x_0 \in \mathbb{R}^2$ there holds*

$$E_R(\mathbf{n}(\cdot, t); x_0) \leq E_{2R}(\mathbf{n}_0(\cdot); x_0) + C_0 \frac{t}{R^2} E_0,$$

where C_0 is an absolute constant independent of t, R and \mathbf{n} .

Proof: Let $\phi(x)$ be a smooth cut-off function satisfying $\phi(x) = 1$ for $x \in B_R(x_0)$ and $\phi(x) = 0$ when $|x - x_0| > 2R$. Multiply $\partial_t \mathbf{n} \phi^2$ on both sides of (1.3), then we have

$$\int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 \phi^2 dx = \alpha \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot \mathbf{h} \phi^2 dx + \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) \phi^2 dx,$$

and using the following relation

$$\frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \phi^2(x) dx = \int_{\mathbb{R}^2} (-\mathbf{h}) \cdot \partial_t \mathbf{n} \phi^2 dx - 2 \int_{\mathbb{R}^2} W_{P_i^j}(\mathbf{n}, \nabla \mathbf{n}) \partial_t \mathbf{n}^j \partial_i \phi \phi dx,$$

hence we get

$$\begin{aligned} & \int_{\mathbb{R}^2} |\partial_t \mathbf{n}|^2 \phi^2 dx + \alpha \frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \phi^2(x) dx \\ & \leq 2\alpha \left| \int_{\mathbb{R}^2} W_{P_i^j}(\mathbf{n}, \nabla \mathbf{n}) \partial_t \mathbf{n}^j \partial_i \phi \phi dx \right| + \beta \int_{\mathbb{R}^2} \partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h}) \phi^2 dx, \end{aligned}$$

and using the equality of (2.3) for the term $\partial_t \mathbf{n} \cdot (\mathbf{n} \times \mathbf{h})$ again, there holds

$$\frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(x, t) \phi^2(x) dx \leq C(\alpha) \int_{\mathbb{R}^2} |\nabla \mathbf{n}|^2 |\nabla \phi|^2 dx \leq C_0 \frac{1}{R^2} E_0.$$

Then the proof is complete. \square

Lemma 2.4 (The positive diffusion) Assume that \mathbf{n} is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2)$. Then there exists $\epsilon_1 > 0$, such that for all $R \in (0, R_0]$ with $R_0 > 0$, if

$$\text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla \mathbf{n}(\cdot, t)|^2 dx < \epsilon_1,$$

then there hold

$$\int_{\mathbb{R}^2 \times [0, T]} |\nabla^2 \mathbf{n}|^2 dx dt \leq C(1 + TR^{-2})E_0, \quad (2.4)$$

and

$$\int_{\mathbb{R}^2 \times [0, T]} |\nabla \mathbf{n}|^4 dx dt \leq C\epsilon_1(1 + TR^{-2})E_0. \quad (2.5)$$

Proof: Due to the embedding inequality (2.1), it suffices to prove the first inequality (2.4). Since

$$\frac{d}{dt} \int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n}) dx = \int_{\mathbb{R}^2} (W_{n^l} - \nabla_i W_{p_i^l}) \cdot n_t^l dx = - \int_{\mathbb{R}^2} \mathbf{h} \cdot \mathbf{n}_t dx,$$

using the equation of (1.3) we have

$$\int_{\mathbb{R}^2} W(\mathbf{n}, \nabla \mathbf{n})(\cdot, t) dx + \alpha \int_0^t \int_{\mathbb{R}^2} (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) \cdot \mathbf{h} dx ds \leq E_0.$$

Next we prove the positivity of the diffusion term. Using Lemma 1.1 and $\mathbf{n} \cdot \Delta \mathbf{n} = -|\nabla \mathbf{n}|^2$, we derive that

$$\begin{aligned} & \int_{\mathbb{R}^2 \times [0, T]} (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) \cdot \mathbf{h} dx dt \\ & \geq \int_{\mathbb{R}^2 \times [0, T]} (\mathbf{n} \times (\nabla_i W_{p_i^l} \times \mathbf{n})) \nabla_i W_{p_i^l} dx dt - C \int_{\mathbb{R}^2 \times [0, T]} |\nabla \mathbf{n}|^2 (|\nabla^2 \mathbf{n}| + |\nabla \mathbf{n}|^2) dx dt \\ & \geq 2a \int_{\mathbb{R}^2 \times [0, T]} \nabla_i W_{p_i^l} \cdot \Delta \mathbf{n} dx dt + 2a \int_{\mathbb{R}^2 \times [0, T]} \Delta \mathbf{n} \cdot (\nabla_i W_{p_i^l} - 2a \Delta \mathbf{n}) dx dt \\ & \quad + \int_{\mathbb{R}^2 \times [0, T]} (\mathbf{n} \times ((\nabla_i W_{p_i^l} - 2a \Delta \mathbf{n}) \times \mathbf{n})) \cdot (\nabla_i W_{p_i^l} - 2a \Delta \mathbf{n}) dx dt \\ & \quad - C \int_{\mathbb{R}^2 \times [0, T]} |\nabla \mathbf{n}|^2 (|\nabla^2 \mathbf{n}| + |\nabla \mathbf{n}|^2) dx dt \\ & \geq 4a \int_{\mathbb{R}^2 \times [0, T]} [a|\Delta \mathbf{n}|^2 + 2(k_1 - a)|\nabla \text{div} \mathbf{n}|^2 + 2(k_2 - a)|\nabla(\nabla \times \mathbf{n} \times \mathbf{n})|^2 \\ & \quad + 2(k_3 - a)|\nabla(\nabla \times \mathbf{n} \cdot \mathbf{n})|^2] dx dt - C \int_{\mathbb{R}^2 \times [0, T]} |\nabla \mathbf{n}|^2 (|\nabla^2 \mathbf{n}| + |\nabla \mathbf{n}|^2) dx dt, \\ & \geq 3a^2 \int_{\mathbb{R}^2 \times [0, T]} |\Delta \mathbf{n}|^2 dx dt - C \int_{\mathbb{R}^2 \times [0, T]} |\nabla \mathbf{n}|^4 dx dt, \end{aligned}$$

and the first estimate (2.4) follows from the embedding inequality (2.1) by choosing a small ϵ_1 . \square

Concluding the above local monotonicity inequality in Lemma 2.3 and the positive diffusion in Lemma 2.4, we have the following corollary.

Corollary 2.5 *Assume that \mathbf{n} is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2)$. Then, there exists $R > 0$ such that $\sup_{x \in \mathbb{R}^2} E_{2R}(\mathbf{n}_0(\cdot); x) \leq \frac{\epsilon_1}{2}$, and*

$$\int_{\mathbb{R}^2 \times [0, t]} |\nabla \mathbf{n}|^4 dx dt + \int_{\mathbb{R}^2 \times [0, t]} |\nabla^2 \mathbf{n}|^2 dx dt \leq C(E_0 + \epsilon_1), \quad (2.6)$$

hold for $t < \frac{\epsilon_1 R^2}{2C_0 E_0}$, where C_0 is given in Lemma 2.3.

Next, we use the idea of Lemma 2.4 and the estimates in Corollary 2.5 to obtain a higher interior regularity of the solution.

Lemma 2.6 *Assume that \mathbf{n} is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2)$. Then there is a constant ϵ_1 such that for all $R \in (0, R_0]$, if*

$$\text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla \mathbf{n}(\cdot, t)|^2 dx < \epsilon_1,$$

then, for all $t \in (\tau, T)$ with $\tau \in (0, T)$, it holds that

$$\int_{\mathbb{R}^2} |\nabla^2 \mathbf{n}(\cdot, t)|^2 dx + \int_{\tau}^t \int_{\mathbb{R}^2} |\nabla^3 \mathbf{n}(\cdot, s)|^2 dx ds \leq C(\epsilon_1, E_0, \tau, T, \frac{T}{R^2}).$$

Proof: First, we can differentiate ∇_{β} to (1.3), multiply it by $\nabla_i \mathbf{h}$ ($i = 1, 2$), and we get

$$\begin{aligned} & \frac{d}{dt} \int_{\mathbb{R}^2} a |\Delta \mathbf{n}|^2 + (k_1 - a) |\nabla \text{div} \mathbf{n}|^2 + (k_2 - a) |\nabla(\mathbf{n} \times (\nabla \times \mathbf{n}))|^2 dx \\ & + \frac{d}{dt} \int_{\mathbb{R}^2} (k_3 - a) |\nabla(\mathbf{n} \cdot (\nabla \times \mathbf{n}))|^2 dx \\ & \leq -\alpha \int_{\mathbb{R}^2} \nabla_i (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx - \beta \int_{\mathbb{R}^2} \nabla_i (\mathbf{n} \times \mathbf{h}) \cdot \nabla_i \mathbf{h} dx \\ & + C \int_{\mathbb{R}^2} [|\nabla \mathbf{n}_t| |\nabla \mathbf{n}| |\nabla^2 \mathbf{n}| + |\mathbf{n}_t| |\nabla^2 \mathbf{n}|^2] dx \\ & \leq -\alpha \int_{\mathbb{R}^2} (\mathbf{n} \times (\nabla_i \mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx \\ & + C \int_{\mathbb{R}^2} [|\nabla \mathbf{n}_t| |\nabla \mathbf{n}| |\nabla^2 \mathbf{n}| + |\mathbf{n}_t| |\nabla^2 \mathbf{n}| |\nabla \mathbf{n}|^2 + |\nabla^2 \mathbf{n}|^2 |\nabla \mathbf{n}|^2 + |\nabla \mathbf{n}| |\nabla^3 \mathbf{n}| |\nabla^2 \mathbf{n}|] dx \\ & \leq -\alpha \int_{\mathbb{R}^2} (\mathbf{n} \times (\nabla_i \mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx + C \int_{\mathbb{R}^2} [|\nabla^2 \mathbf{n}|^2 |\nabla \mathbf{n}|^2 + |\nabla \mathbf{n}| |\nabla^3 \mathbf{n}| |\nabla^2 \mathbf{n}|] dx \end{aligned} \quad (2.7)$$

Note the fact that $|\mathbf{n} \cdot \nabla_i \Delta \mathbf{n}| \leq C |\nabla \mathbf{n}| |\nabla^2 \mathbf{n}|$, and similar estimates as in Lemma 2.4 imply

$$\int_{\mathbb{R}^2} (\mathbf{n} \times (\nabla_i \mathbf{h} \times \mathbf{n})) \cdot \nabla_i \mathbf{h} dx \geq 3a^2 \int_{\mathbb{R}^2} |\nabla^3 \mathbf{n}|^2 - C \int_{\mathbb{R}^2} |\nabla \mathbf{n}|^2 |\nabla^2 \mathbf{n}|^2 dx.$$

Due to the interpolation inequality

$$\|\nabla^2 \mathbf{n}\|_4 \leq C \|\nabla^2 \mathbf{n}\|_2^{1/2} \|\nabla^3 \mathbf{n}\|_2^{1/2},$$

we have

$$\int_{\mathbb{R}^2} |\nabla \mathbf{n}|^2 |\nabla^2 \mathbf{n}|^2 dx \leq \delta \|\nabla^3 \mathbf{n}\|_2^2 + C(\delta) \|\nabla \mathbf{n}\|_4^4 \|\nabla^2 \mathbf{n}\|_2^2,$$

thus Gronwall's inequality and Corollary 2.5 imply the required estimates. \square

Indeed, using the above idea by induction, one can prove the smooth property of \mathbf{n} , and we omit the proof (similar arguments for Ericksen-Leslie system, see [36, Corollary 4.6]).

Corollary 2.7 *Assume that \mathbf{n} is a smooth solution of the Landau-Lifshitz equation (1.3) in $(0, T) \times \mathbb{R}^2$ and the initial data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2)$. Then there is a constant $\epsilon_1 > 0$ such that for all $R \in (0, R_0]$, if*

$$\text{esssup}_{0 \leq t \leq T, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla \mathbf{n}(\cdot, t)|^2 dx < \epsilon_1,$$

then, for all $t \in (\tau, T)$ with $\tau \in (0, T)$, for any $l \geq 1$ it holds that

$$\begin{aligned} & \int_{\mathbb{R}^2} |\nabla^{l+1} \mathbf{n}(\cdot, t)|^2 dx + \int_{\tau}^t \int_{\mathbb{R}^2} |\nabla^{l+2} \mathbf{n}(\cdot, s)|^2 dx ds \\ & \leq C(l, \epsilon_1, E_0, \tau, T, \frac{T}{R^2}). \end{aligned} \quad (2.8)$$

Moreover, \mathbf{n} is regular for all $t \in (0, T)$.

2.2 Existence of global weak solution

Now we complete the proof of the existence part in Theorem 1.2. Similar to [32, 24, 36], we sketch its step for completeness.

For any data $\mathbf{n}_0 \in H_b^1(\mathbb{R}^2; S^2)$, one can approximate it by a sequence of smooth maps \mathbf{n}_0^k in $H_b^1(\mathbb{R}^2; S^2)$, and we can assume that $\nabla \mathbf{n}_0^k \in H_b^4(\mathbb{R}^2; S^2)$ (see [31]). Due to the absolute continuity property of the integral, for any $\epsilon_1 > 0$, there exists $R_0 \geq R_1 > 0$ such that

$$\sup_{x \in \mathbb{R}^2} \int_{B_{R_1}(x)} |\nabla \mathbf{n}_0|^2 dx \leq \epsilon_1,$$

and by the strong convergence of \mathbf{n}_0^k ,

$$\sup_{x \in \mathbb{R}^2} \int_{B_{R_1}(x)} |\nabla \mathbf{n}_0^k|^2 dx \leq 2\epsilon_1$$

for a sufficient large k . Without loss of generality, we assume that it holds for all $k \geq 1$.

For the data \mathbf{n}_0^k , by Theorem A.1 there exists a time T^k and a strong solution \mathbf{n}^k such that

$$\nabla \mathbf{n}^k \in C([0, T^k]; H^4(\mathbb{R}^2)).$$

Hence there exists $T_0^k \leq T^k$ such that

$$\sup_{0 < t < T_0^k, x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla \mathbf{n}^k(y, t)|^2 dy \leq (8 + \frac{1}{a}) \epsilon_1,$$

where $R \leq R_1/2$. However, by the local monotonic inequality in Lemma 2.3, we have $T_0^k \geq \frac{\epsilon_1 R_1^2}{4C_0 E_0} = T_0 > 0$ uniformly. For any $0 < \tau < T_0$, by the estimates in Corollary 2.7 for any $l \geq 1$ we get

$$\sup_{\tau < t < T_0} \int_{\mathbb{R}^2} |\nabla^{l+1} \mathbf{n}^k|^2(\cdot, t) dx + \int_{\tau}^{T_0} \int_{\mathbb{R}^2} |\nabla^{l+2} \mathbf{n}^k(\cdot, s)|^2 dx ds \leq C(l, \epsilon_1, E_0, \tau, T_0, \frac{T_0}{R^2}). \quad (2.9)$$

Moreover, the energy inequality in Lemma 2.2, *a priori* estimates in Lemma 2.4 and the equation (1.3) yield that

$$E(\mathbf{n}^k)(t) \leq E_0, \quad 0 < t < T^k, \quad (2.10)$$

and

$$\int_{\mathbb{R}^2 \times [0, T_0^k]} (|\nabla^2 \mathbf{n}^k|^2 + |\partial_t \mathbf{n}^k|^2 + |\nabla \mathbf{n}^k|^4) dx dt \leq C(\epsilon_1, C_0, E_0). \quad (2.11)$$

Hence the above estimates (2.9)-(2.11) and Aubin-Lions Lemma yield that there exists a solution $\mathbf{n} - b \in W_2^{2,1}(\mathbb{R}^2 \times [0, T_0]; \mathbb{R}^3)$ such that (at most up to a subsequence)

$$\mathbf{n}^k - b \rightarrow \mathbf{n} - b, \quad \text{locally in } W_2^{3,1}(\mathbb{R}^2 \times (0, T_0); \mathbb{R}^3).$$

By (2.10), $\nabla \mathbf{n}(t) \rightharpoonup \nabla \mathbf{n}_0$ weakly in $L^2(\mathbb{R}^2)$, thus $E(\mathbf{n}_0) \leq \liminf_{t \rightarrow 0} E(\mathbf{n}(t))$. On the other hand, by the energy estimates of (\mathbf{n}^k) , we have

$$E(\mathbf{n}_0) \geq \limsup_{t \rightarrow 0} E(\mathbf{n}(t)).$$

Hence, $\nabla \mathbf{n}(t) \rightarrow \nabla \mathbf{n}_0$ strongly in $L^2(\mathbb{R}^2)$ and \mathbf{n} is the solution of the equation (1.3) with the initial data \mathbf{n}_0 . From the weak limit of regular estimates (2.9), we know that $\mathbf{n} \in C^\infty(\mathbb{R}^2 \times (0, T_0])$ and $\nabla^{l+1} \mathbf{n}(\cdot, T_0) \in L^2(\mathbb{R}^2)$ for any $l \geq 1$. By Theorem A.1, there exists a unique smooth solution of (1.3) with the initial data $\mathbf{n}(\cdot, T_0)$, which is still written as \mathbf{n} , and blow-up criterion yields that if \mathbf{n} blows up at finite time T^* , then

$$\|\nabla \mathbf{n}\|_{L^\infty(\mathbb{R}^2)}(t) \rightarrow \infty, \quad \text{as } t \rightarrow T^*.$$

As a result, we have

$$|\nabla^4 \mathbf{n}|(x, t) \notin L_t^\infty L_x^2((T_0, T^*) \times \mathbb{R}^2) \quad (2.12)$$

We assume that T_1 is the first singular time of \mathbf{n} , then we have

$$\mathbf{n} \in C^\infty(\mathbb{R}^2 \times (0, T_1); S^2) \quad \text{and} \quad \mathbf{n} \notin C^\infty(\mathbb{R}^2 \times (0, T_1]; S^2);$$

and by Corollary 2.7 and (2.12), there exists $\epsilon_0 > 0$ such that

$$\limsup_{t \uparrow T_1} \sup_{x \in \mathbb{R}^2} \int_{B_R(x)} |\nabla \mathbf{n}|^2(\cdot, t) \geq \epsilon_0, \quad \forall R > 0.$$

Finally, since $\mathbf{n} - b \in C^0([0, T_1], L^2(\mathbb{R}^2))$ by the interpolation inequality (similarly see P330, [24]), we can define

$$\mathbf{n}(T_1) - b = \lim_{t \uparrow T_1} \mathbf{n}(t) - b \quad \text{in } L^2(\mathbb{R}^2).$$

On the other hand, by the energy inequality $\nabla \mathbf{n} \in L^\infty(0, T_1; L^2(\mathbb{R}^2))$, hence $\nabla \mathbf{n}(t) \rightharpoonup \nabla \mathbf{n}(T_1)$. Similarly we can extend T_1 to T_2 and so on. It's easy to check that the energy loss at every singular time T_i for $i \geq 1$ is at least ϵ_1 , thus the number L of the singular time is finite. Moreover, singular points at every singular time are finite by similar arguments as in [32], since $\partial_t u \in L^2_{x,t}$ in Lemma 2.2 and the local monotonicity inequality in Lemma 2.3 hold. Assume that singular points are (x_i^j, T_i) with $1 \leq j \leq L_i$ and $i \leq L$, and we have

$$\limsup_{t \uparrow T_i} \int_{B_R(x_i^j)} |\nabla \mathbf{n}|^2(\cdot, t) \geq \epsilon_0, \quad \forall R > 0.$$

The proof is complete. \square

3 Uniqueness of weak solution

In this section, we follow the same route as in [37] and prove the following uniqueness theorem. The main difference is to deal with the Schrödinger part $\mathbf{n} \times \mathbf{h}$.

Theorem 3.1 *Let \mathbf{n}^1 and \mathbf{n}^2 be two weak solutions of the Landau-Lifshitz equation (1.3) in \mathbb{R}^2 obtained in Theorem 1.2 with the same initial data \mathbf{n}_0 . Then we have*

$$\mathbf{n}^1(t) = \mathbf{n}^2(t)$$

for any $t \in [0, +\infty)$.

Let \mathbf{n}^1 and \mathbf{n}^2 be two weak solutions of the Landau-Lifshitz equation (1.3) in \mathbb{R}^2 obtained in Theorem 1.2 with the same initial data \mathbf{n}_0 . Let

$$\delta_{\mathbf{n}} = \mathbf{n}^2 - \mathbf{n}^1,$$

then we infer that

$$\partial_t \delta_{\mathbf{n}} = \alpha \delta_{\mathbf{n} \times (\mathbf{h} \times \mathbf{n})} + \beta \delta_{\mathbf{n} \times \mathbf{h}}. \quad (3.1)$$

Here and in what follows, we denote $f^i = f(\mathbf{n}^i)$ for $i = 1, 2$ and $\delta_f = f^2 - f^1$ if f is a function of \mathbf{n} .

Different with [32, 18], it's not easy to obtain the positivity of the diffusion term $\nabla \delta_{\mathbf{n}}$ under the metric of L^2 norm, since we can't use the property of $\Delta \mathbf{n} \cdot \mathbf{n} = -|\nabla \mathbf{n}|^2$ from $|\mathbf{n}| = 1$. Instead, we introduce a type of weak Oseen-Frank metric

$$W(t) = \sup_{j \geq 0} 2^{-2js} \int_{\mathbb{R}^2} W^j(t, x) dx + \|\Delta_{-1} \delta_{\mathbf{n}}\|_2^2$$

with $s \in (0, 1)$ and

$$\begin{aligned} W^j(x, t) = & a |\nabla \Delta_j \delta_{\mathbf{n}}|^2 + (k_1 - a) |\operatorname{div} \Delta_j \delta_{\mathbf{n}}|^2 \\ & + (k_2 - a) |\mathbf{n}^2 \times (\nabla \times \Delta_j \delta_{\mathbf{n}})|^2 + (k_3 - a) |\mathbf{n}^2 \cdot (\nabla \times \Delta_j \delta_{\mathbf{n}})|^2. \end{aligned}$$

The proof of Theorem 3.1 is based on the following two propositions. To state them, we introduce

$$\bar{h}(t) \stackrel{\text{def}}{=} 1 + \|(\nabla \mathbf{n}^1, \nabla \mathbf{n}^2)\|_4^4 + \|(\partial_t \mathbf{n}^1, \partial_t \mathbf{n}^2)\|_2^2 + \|(\nabla \mathbf{n}^1, \nabla \mathbf{n}^2)\|_{H^1}^2.$$

Proposition 3.2 *It holds that*

$$\frac{d}{dt} \|\Delta_{-1} \delta_{\mathbf{n}}\|_2^2 \leq C \bar{h}(t) W(t).$$

Proposition 3.3 *For any $j \geq 0$ and $\epsilon > 0$, it holds that*

$$\frac{d}{dt} \int_{\mathbb{R}^2} W^j(x, t) dx + 3\alpha a^2 \|\Delta_j \nabla^2 \delta_{\mathbf{n}}\|_2^2 \leq C 2^{2js} \bar{h}(t) W(t) + \epsilon \sum_{l=j-9}^{j+9} 2^{4l} \|\Delta_l \delta_{\mathbf{n}}\|_2^2.$$

For the moment, let us assume that these propositions are correct and complete the proof of Theorem 3.1. Assume that T_1^i is the first blow-up time of \mathbf{n}^i with $i = 1, 2$. We know from Lemma 2.4 that

$$\int_{\mathbb{R}^2 \times [0, T_1 - \theta]} |\nabla^2 \mathbf{n}^i|^2 + |\nabla \mathbf{n}^i|^4 dx dt < +\infty, \quad (3.2)$$

where $\theta > 0$ and $T_1 = \min\{T_1^1, T_1^2\}$. And using the equation (1.3), we get

$$\partial_t \mathbf{n}^i \in L^2((0, T_1 - \theta) \times \mathbb{R}^2). \quad (3.3)$$

Proposition 3.2 and Proposition 3.3 ensure that

$$\frac{d}{dt} \left(\int_{\mathbb{R}^2} W^j dx + \|\Delta_{-1} \delta_{\mathbf{n}}\|_2^2 \right) + ca 2^{4j} \|\Delta_j \delta_{\mathbf{n}}\|_2^2 \leq C 2^{2js} \bar{h}(t) W(t) + \epsilon \sum_{l=j-9}^{j+9} 2^{4l} \|\Delta_l \delta_{\mathbf{n}}\|_2^2.$$

Noting that $\int_{\mathbb{R}^2} W^j dx + \|\Delta_{-1} \delta_{\mathbf{n}}\|_2^2 \geq c 2^j \|\Delta_j \delta_{\mathbf{n}}\|_2^2$, we deduce by taking ϵ small enough that

$$W(t) \leq C \int_0^t \bar{h}(\tau) W(\tau) d\tau.$$

By (3.2) and (3.3), $\bar{h}(t) \in L^1(0, T_1 - \theta)$. Then by Gronwall's inequality, we get $W(t) = 0$ for $t \in [0, T_1 - \theta]$ for any $\theta > 0$. Hence, $\mathbf{n}^1(t) = \mathbf{n}^2(t)$ on $[0, T_1)$ with $T_1 > 0$ the first singular time of the solution \mathbf{n}^1 or \mathbf{n}^2 . Since $\mathbf{n}^i \in C_w([0, +\infty); H_b^1)$, $\mathbf{n}^1(T_1) = \mathbf{n}^2(T_1)$. Then the same arguments show that there exists a $T_2 > T_1$ such that $\mathbf{n}^1(t) = \mathbf{n}^2(t)$ on $[T_1, T_2)$, where T_2 is the second singular time of the solution \mathbf{n}^1 or \mathbf{n}^2 . Since the number of singular time is finite, we can conclude that $\mathbf{n}^1(t) = \mathbf{n}^2(t)$ for $t \in [0, +\infty)$. \square

3.1 Littlewood-Paley theory and nonlinear estimates

Let us recall some basic facts on Littlewood-Paley theory (see [8] for more details). Choose two nonnegative radial functions $\chi, \phi \in \mathcal{S}(\mathbb{R}^n)$ supported respectively in $\{\xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3}\}$ and $\{\xi \in \mathbb{R}^n, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that for any $\xi \in \mathbb{R}^n$,

$$\chi(\xi) + \sum_{j \geq 0} \phi(2^{-j}\xi) = 1.$$

The frequency localization operator Δ_j and S_j are defined by

$$\begin{aligned} \Delta_j f &= \phi(2^{-j}D)f = 2^{nj} \int_{\mathbb{R}^n} h(2^j y) f(x-y) dy, \quad \text{for } j \geq 0, \\ S_j f &= \chi(2^{-j}D)f = \sum_{-1 \leq k \leq j-1} \Delta_k f = 2^{nj} \int_{\mathbb{R}^n} \tilde{h}(2^j y) f(x-y) dy, \\ \Delta_{-1} f &= S_0 f, \quad \Delta_j f = 0 \quad \text{for } j \leq -2, \end{aligned}$$

where $h = \mathcal{F}^{-1}\phi$, $\tilde{h} = \mathcal{F}^{-1}\chi$. With this choice of ϕ , it is easy to verify that

$$\Delta_j \Delta_k f = 0, \quad \text{if } |j-k| \geq 2; \quad \Delta_j (S_{k-1} f \Delta_k f) = 0, \quad \text{if } |j-k| \geq 5. \quad (3.4)$$

In terms of Δ_j , the norm of the inhomogeneous Besov space $B_{p,q}^s$ for $s \in \mathbb{R}$, and $p, q \geq 1$ is defined by

$$\|f\|_{B_{p,q}^s} \stackrel{\text{def}}{=} \left\| \{2^{js} \|\Delta_j f\|_p\}_{j \geq -1} \right\|_{\ell^q},$$

and

$$\|f\|_{B_{p,\infty}^s} \stackrel{\text{def}}{=} \sup_{j \geq -1} \{2^{js} \|\Delta_j f\|_p\}.$$

We will constantly use the following Bernstein's inequality [8].

Lemma 3.4 *Let $c \in (0, 1)$ and $R > 0$. Assume that $1 \leq p \leq q \leq \infty$ and $f \in L^p(\mathbb{R}^n)$. Then*

$$\begin{aligned} \text{supp } \hat{f} \subset \{|\xi| \leq R\} &\Rightarrow \|\partial^\alpha f\|_q \leq C R^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})} \|f\|_p, \\ \text{supp } \hat{f} \subset \{cR \leq |\xi| \leq R\} &\Rightarrow \|f\|_p \leq C R^{-|\alpha|} \sup_{|\beta|=|\alpha|} \|\partial^\beta f\|_p, \end{aligned}$$

where the constant C is independent of f and R .

We need the following nonlinear estimates, seeing [37] for more details.

Lemma 3.5 *Let $s \in (0, 1)$. For any $j \geq -1$, we have*

$$\|\Delta_j(fgh)\|_2 \leq C 2^{js} (\|f\|_\infty + \|\nabla f\|_2) \|g\|_{B_{2,\infty}^{1-s}} \|h\|_2.$$

Lemma 3.6 *Let $s \in (0, 1)$. For any $j \geq -1$, we have*

$$\begin{aligned} \|\Delta_j(f \nabla gh)\|_2 &\leq C 2^{js} \|g\|_{B_{2,\infty}^{1-s}} (\|f\|_\infty \|h\|_{H^1} + \|\nabla f\|_4 \|h\|_4 + \|\nabla^2 f\|_2 \|h\|_2) \\ &\quad + C 2^{\frac{js}{2}} \|f\|_\infty \|h\|_4 \|g\|_{B_{2,\infty}^{1-s}}^{\frac{1}{2}} \sum_{l=j-9}^{j+9} 2^{\frac{l}{2}} \|\Delta_l \nabla g\|_{\frac{1}{2}}^{\frac{1}{2}}. \end{aligned}$$

Lemma 3.7 *Let $s \in (0, 1)$. For any $j \geq -1$, it holds that*

$$\|[\Delta_j, f]\nabla g\|_2 \leq C2^{\frac{js}{2}}\|\nabla f\|_4\|g\|_{B_{2,\infty}^{-s}}^{\frac{1}{2}} \sum_{|j'-j|\leq 4} 2^{\frac{j'}{2}}\|\Delta_{j'}g\|_2^{\frac{1}{2}} + C2^{js}\|g\|_{B_{2,\infty}^{-s}}(\|f\|_\infty + \|\nabla^2 f\|_2).$$

3.2 Proof of Proposition 3.2

Using the equation (3.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta_{-1}\delta_{\mathbf{n}}\|_2^2 = \alpha\langle\Delta_{-1}\delta_{\mathbf{n}\times(\mathbf{h}\times\mathbf{n})}, \Delta_{-1}\delta_{\mathbf{n}}\rangle + \beta\langle\Delta_{-1}\delta_{\mathbf{n}\times\mathbf{h}}, \Delta_{-1}\delta_{\mathbf{n}}\rangle \triangleq I.$$

Recall that the formula of \mathbf{h} in (1.4), and we could write I as

$$\begin{aligned} I &= \langle\Delta_{-1}(\mathcal{M}\nabla^2\delta_{\mathbf{n}}), \Delta_{-1}\delta_{\mathbf{n}}\rangle + \sum_{i=1,2} \langle\Delta_{-1}(\mathcal{M}\nabla^2\mathbf{n}^i\delta_{\mathbf{n}}), \Delta_{-1}\delta_{\mathbf{n}}\rangle \\ &\quad + \sum_{i=1,2} \langle\Delta_{-1}(\mathcal{M}\nabla\mathbf{n}^i\delta_{\nabla\mathbf{n}}), \Delta_{-1}\delta_{\mathbf{n}}\rangle + \sum_{i,k=1,2} \langle\Delta_{-1}(\mathcal{M}\nabla\mathbf{n}^i\nabla\mathbf{n}^k\delta_{\mathbf{n}}), \Delta_{-1}\delta_{\mathbf{n}}\rangle. \\ &= I_1 + \dots + I_4. \end{aligned}$$

Here and in what follows, we denote by \mathcal{M} a polynomial function of $(\mathbf{n}^1, \mathbf{n}^2)$ with degree no greater than 4, which may be different from line to line. Then by Lemma 3.5 (for I_2, I_4), Lemma 3.6 and Lemma 3.4 (for I_1, I_3), we get

$$|I| \leq C\bar{h}(t)W(t).$$

Thus the proof is complete. \square

3.3 Proof of Proposition 3.3

Let us first derive the following evolution inequality for the Oseen-Frank density.

Lemma 3.8 *For any $j \geq 0$, it holds that*

$$\frac{d}{dt} \int_{\mathbb{R}^2} W^j(t, x) dx + 3\alpha a^2 \|\Delta_j \nabla^2 \delta_{\mathbf{n}}\|_2^2 \leq B_1 + \dots + B_6,$$

where B_i will be given in the proof.

The key part of the above lemma is the positivity of the diffusion term $\mathbf{n} \times (\Delta_j \delta_{\mathbf{h}} \times \mathbf{n}) \cdot \Delta_j \delta_{\mathbf{h}}$. It's important to analysis the main parts of $\Delta_j \delta_{\mathbf{h}}$ and $\Delta_j \delta_{\mathbf{n}\times(\mathbf{h}\times\mathbf{n})}$ (the second derivative terms). Using $\mathbf{curl}(fu) = f\mathbf{curl}u + \nabla f \times u$, \mathbf{h} in (1.4) can be rewritten as

$$\begin{aligned} \mathbf{h} &= 2a\Delta\mathbf{n} + 2(k_1 - a)\nabla\text{div}\mathbf{n} - 2(k_2 - a)\mathbf{curl}\mathbf{curl}\mathbf{n} - 2(k_3 - k_2)(\nabla\mathbf{curl}\mathbf{n} \cdot \mathbf{n}) \times \mathbf{n} \\ &\quad - 2(k_3 - k_2)(2(\mathbf{n} \cdot \mathbf{curl}\mathbf{n})\mathbf{curl}\mathbf{n} + (\nabla\mathbf{n} \cdot \mathbf{curl}\mathbf{n}) \times \mathbf{n}), \end{aligned} \quad (3.5)$$

hence the main parts of $\Delta_j \delta_{\mathbf{h}}$ is

$$\begin{aligned} W_1 &= 2a\Delta\Delta_j\delta_{\mathbf{n}} + 2(k_1 - a)\nabla\text{div}\Delta_j\delta_{\mathbf{n}} - 2(k_2 - a)\mathbf{curl}\mathbf{curl}\Delta_j\delta_{\mathbf{n}} \\ &\quad - 2(k_3 - k_2)(\nabla\mathbf{curl}\Delta_j\delta_{\mathbf{n}} \cdot \mathbf{n}^2) \times \mathbf{n}^2. \end{aligned} \quad (3.6)$$

Note that by (3.5)

$$\begin{aligned}
& (\mathbf{h} \cdot \mathbf{n})\mathbf{n} \\
&= (-2a|\nabla \mathbf{n}|^2 + 2(k_1 - a)\mathbf{n} \cdot \nabla \operatorname{div} \mathbf{n} - 2(k_2 - a)\mathbf{n} \cdot \operatorname{curl} \operatorname{curl} \mathbf{n} - 4(k_3 - k_2)(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2) \mathbf{n} \\
&= 2(k_1 - a)(\mathbf{n} \cdot \nabla \operatorname{div} \mathbf{n})\mathbf{n} - 2(k_2 - a)(\mathbf{n} \cdot \operatorname{curl} \operatorname{curl} \mathbf{n})\mathbf{n} \\
&\quad - 2a|\nabla \mathbf{n}|^2 \mathbf{n} - 4(k_3 - k_2)(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2 \mathbf{n},
\end{aligned}$$

and $\mathbf{n} \times (\mathbf{h} \times \mathbf{n}) = \mathbf{h} - (\mathbf{h} \cdot \mathbf{n})\mathbf{n}$. We deduce

$$\begin{aligned}
& \delta_{\mathbf{n} \times (\mathbf{h} \times \mathbf{n})} \\
&= 2a\Delta \delta_{\mathbf{n}} + 2(k_1 - a)\nabla \operatorname{div} \delta_{\mathbf{n}} - 2(k_2 - a)\operatorname{curl} \operatorname{curl} \delta_{\mathbf{n}} - 2(k_3 - k_2)(\nabla \operatorname{curl} \delta_{\mathbf{n}} \cdot \mathbf{n}^2) \times \mathbf{n}^2 \\
&\quad - 2(k_1 - a)(\mathbf{n}^2 \cdot \nabla \operatorname{div} \delta_{\mathbf{n}})\mathbf{n}^2 + 2(k_2 - a)(\mathbf{n}^2 \cdot \operatorname{curl} \operatorname{curl} \delta_{\mathbf{n}})\mathbf{n}^2 \\
&\quad + \sum_{i=1,2} (\mathcal{M} \delta_{\mathbf{n}} \nabla^2 \mathbf{n}^i + \mathcal{M} \delta_{\nabla \mathbf{n}} \nabla \mathbf{n}^i) + \sum_{i,k=1,2} \mathcal{M} \nabla \mathbf{n}^i \nabla \mathbf{n}^k \delta_{\mathbf{n}}.
\end{aligned} \tag{3.7}$$

Denote the main parts of $\Delta_j \delta_{\mathbf{n} \times (\mathbf{h} \times \mathbf{n})}$ as follows.

$$\begin{aligned}
H_1 &= 2a\Delta \Delta_j \delta_{\mathbf{n}} + 2(k_1 - a)\nabla \operatorname{div} \Delta_j \delta_{\mathbf{n}} - 2(k_2 - a)\operatorname{curl} \operatorname{curl} \Delta_j \delta_{\mathbf{n}} \\
&\quad - 2(k_3 - k_2)(\nabla \operatorname{curl} \Delta_j \delta_{\mathbf{n}} \cdot \mathbf{n}^2) \times \mathbf{n}^2 - 2(k_1 - a)(\mathbf{n}^2 \cdot \nabla \operatorname{div} \Delta_j \delta_{\mathbf{n}})\mathbf{n}^2 \\
&\quad + 2(k_2 - a)(\mathbf{n}^2 \cdot \operatorname{curl} \operatorname{curl} \Delta_j \delta_{\mathbf{n}})\mathbf{n}^2.
\end{aligned} \tag{3.8}$$

Lemma 3.9 Assume that W_1, H_1 state as above, then we have

$$\frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx \geq \frac{3}{4} a^2 \|\Delta \Delta_j \delta_{\mathbf{n}}\|_2^2 - B_1.$$

where

$$B_1 = |\langle \mathcal{M} \nabla \mathbf{n}^2 \Delta_j \nabla^2 \delta_{\mathbf{n}}, \Delta_j \nabla \delta_{\mathbf{n}} \rangle|.$$

Proof of Lemma 3.9. Let $S_1 = \Delta_j \Delta \delta_{\mathbf{n}}$, and

$$H_2 = (k_1 - a)\nabla \operatorname{div} \Delta_j \delta_{\mathbf{n}} - (k_2 - a)\operatorname{curl} \operatorname{curl} \Delta_j \delta_{\mathbf{n}} - (k_3 - k_2)(\nabla \operatorname{curl} \Delta_j \delta_{\mathbf{n}} \cdot \mathbf{n}^2) \times \mathbf{n}^2.$$

Then we find

$$\begin{aligned}
\frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx &= \int_{\mathbb{R}^2} (aS_1 + H_2) \cdot (aS_1 + H_2 - (\mathbf{n}^2 \cdot H_2)\mathbf{n}^2) dx \\
&= a^2 \|S_1\|_2^2 + a \langle H_2, S_1 \rangle + \|H_2 \times \mathbf{n}^2\|_2^2 + a \langle S_1, \mathbf{n}^2 \times (H_2 \times \mathbf{n}^2) \rangle \\
&\quad + \frac{a^2}{4} \|\mathbf{n}^2 \times S_1\|_2^2 - \frac{a^2}{4} \|\mathbf{n}^2 \times S_1\|_2^2 \\
&\geq \frac{3}{4} a^2 \|\Delta_j \Delta \delta_{\mathbf{n}}\|_2^2 + a \langle H_2, S_1 \rangle.
\end{aligned}$$

Furthermore, by Lemma A.3 we have

$$\begin{aligned}
\langle H_2, S_1 \rangle &= (k_1 - a) \|\nabla \operatorname{div} \Delta_j \delta_{\mathbf{n}}\|_2^2 + (k_2 - a) \|\nabla \operatorname{curl} \Delta_j \delta_{\mathbf{n}}\|_2^2 \\
&\quad + (k_3 - k_2) \langle \nabla \Delta_j \operatorname{curl} \delta_{\mathbf{n}} \cdot \mathbf{n}^2, \nabla \Delta_j \operatorname{curl} \delta_{\mathbf{n}} \cdot \mathbf{n}^2 \rangle - B_1 \geq -B_1.
\end{aligned}$$

Hence we get

$$\frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx \geq \frac{3}{4} a^2 \|\Delta \Delta_j \delta_{\mathbf{n}}\|_2^2 - B_1.$$

The proof is complete. \square

Proof of Lemma 3.8. Due to the definition of W^j , we have

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} W^j(t, x) dx &= \int_{\mathbb{R}^2} -\nabla_i W_{p_i^k}^j \partial_t \Delta_j \delta_{n_k} + W_{n_l}^j (n_l^2)_t dx \\ &\triangleq - \int_{\mathbb{R}^2} \nabla_i W_{p_i^k}^j \partial_t \Delta_j \delta_{n_k} dx + B_1. \end{aligned}$$

Using the equation (1.3), we get

$$\begin{aligned} - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \Delta_j \delta_{\mathbf{n}_t} dx &= -\alpha \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \Delta_j (\mathbf{n}^2 \times (\mathbf{h}^2 \times \mathbf{n}^2) - \mathbf{n}^1 \times (\mathbf{h}^1 \times \mathbf{n}^1)) dx \\ &\quad - \beta \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \Delta_j (\mathbf{n}^2 \times \mathbf{h}^2 - \mathbf{n}^1 \times \mathbf{h}^1) dx \\ &\triangleq \alpha I' + \beta I''. \end{aligned}$$

So, we conclude that

$$\frac{d}{dt} \int_{\mathbb{R}^2} W^j(t, x) dx \leq \alpha I' + \beta I'' + B_2. \quad (3.9)$$

As in Lemma 1.1, we have

$$\begin{aligned} \nabla_\alpha W_{p_\alpha^l}^j &= 2a \Delta_j \Delta \delta_{\mathbf{n}} + 2(k_1 - a) \nabla \operatorname{div} \Delta_j \delta_{\mathbf{n}} - 2(k_2 - a) \operatorname{curl} \operatorname{curl} \Delta_j \delta_{\mathbf{n}} \\ &\quad - 2(k_3 - k_2) \operatorname{curl} ((\mathbf{n}^2 \cdot \operatorname{curl} \Delta_j \delta_{\mathbf{n}}) \mathbf{n}^2), \end{aligned}$$

where $p_\alpha^l = \nabla_\alpha \Delta_j (\mathbf{n}^2 - \mathbf{n}^1)_l$, and we get

$$\nabla_\alpha W_{p_\alpha^l}^j = W_1 + \mathcal{M} \nabla \mathbf{n}^2 \nabla \Delta_j \delta_{\mathbf{n}}, \quad (3.10)$$

that is, W_1 is also the main part of $\nabla_\alpha W_{p_\alpha^l}^j$. Then we have

$$\begin{aligned} I'' &= - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \cdot (\mathbf{n}^2 \times \Delta_j \delta_{\mathbf{h}}) dx - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \cdot ([\Delta_j, \mathbf{n}^2 \times] \delta_{\mathbf{h}}) dx \\ &\quad - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \Delta_j (\delta_{\mathbf{n}} \times \mathbf{h}^1) dx \\ &\triangleq B_5 + B_3 + B_4, \end{aligned}$$

where B_5 can be further decomposed into

$$B_5 = - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j) \cdot (\mathbf{n}^2 \times (\Delta_j \delta_{\mathbf{h}} - W_1)) dx - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j - W_1) \cdot (\mathbf{n}^2 \times W_1) dx.$$

On the other hand, for the estimate of I' , we have

$$\begin{aligned} I' &= - \int_{\mathbb{R}^2} W_1 \cdot H_1 dx - \int_{\mathbb{R}^2} (\nabla_i W_{p_i^l}^j - W_1) \cdot H_1 dx \\ &\quad - \int_{\mathbb{R}^2} \nabla_i W_{p_i^l}^j \cdot (\Delta_j \delta_{\mathbf{n} \times \mathbf{h} \times \mathbf{n}} - H_1) dx \\ &\triangleq - \int_{\mathbb{R}^2} W_1 \cdot H_1 dx + B_1 + B_6. \end{aligned}$$

Due to Lemma 3.9,

$$\frac{1}{4} \int_{\mathbb{R}^2} W_1 \cdot H_1 dx \geq \frac{3}{4} a^2 \|\Delta \Delta_j \delta_{\mathbf{n}}\|_2^2 - B_1,$$

which along with (3.9) gives the lemma. \square

Now we follow the same route as in [37] and begin with the estimates of B_i .

• Estimate of B_1 .

By (3.10) and the definition of W_1 and H_1 , we have

$$\begin{aligned} B_5 &\leq C \|\nabla \mathbf{n}^2\|_4 \|\Delta_j \nabla \delta_{\mathbf{n}}\|_4 \|\Delta_j \nabla^2 \delta_{\mathbf{n}}\|_2 \leq C \|\nabla \mathbf{n}^2\|_4 \|\Delta_j \nabla \delta_{\mathbf{n}}\|_2^{\frac{1}{2}} \|\Delta_j \nabla^2 \delta_{\mathbf{n}}\|_2^{\frac{3}{2}} \\ &\leq C \|\nabla \mathbf{n}^2\|_4^4 \|\Delta_j \nabla \delta_{\mathbf{n}}\|_2^2 + \epsilon 2^{4j} \|\Delta_j \delta_{\mathbf{n}}\|_2^2 \\ &\leq \epsilon 2^{4j} \|\Delta_j \delta_{\mathbf{n}}\|_2^2 + C 2^{2js} \bar{h}(t) W(t). \end{aligned}$$

• Estimate of B_2 . Recall that

$$W_{\mathbf{n}_i^2}^j = 2(k_3 - k_2)(\mathbf{n}^2 \cdot \mathbf{curl} \Delta_j \delta_{\mathbf{n}}) \mathbf{curl} \Delta_j \delta_{\mathbf{n}}, \quad (3.11)$$

then Lemma 3.4 yields that

$$\begin{aligned} B_1 &\leq C 2^{2j} \|\Delta_j \delta_{\mathbf{n}}\|_2 \|\partial_t \mathbf{n}^2\|_2 \|\Delta_j \delta_{\mathbf{n}}\|_{\infty} \leq C 2^{3j} \|\Delta_j \delta_{\mathbf{n}}\|_2^2 \|\partial_t \mathbf{n}^2\|_2 \\ &\leq \epsilon 2^{4j} \|\Delta_j \delta_{\mathbf{n}}\|_2^2 + C 2^{2j} \|\partial_t \mathbf{n}^2\|_2^2 \|\Delta_j \delta_{\mathbf{n}}\|_2^2 \\ &\leq \epsilon 2^{4j} \|\Delta_j \delta_{\mathbf{n}}\|_2^2 + C 2^{2js} \bar{h}(t) W(t). \end{aligned}$$

• Estimate of B_6, B_3, B_4, B_5 .

By (3.10) and Lemma 3.4, for $j \geq 0$ we have

$$\|\nabla_i W_{p_i^l}^j\|_2 \leq C (\|\nabla^2 \Delta_j \delta_{\mathbf{n}}\|_2 + \| |\nabla \mathbf{n}^2| \Delta_j \delta_{\mathbf{n}} \|_2) \leq C 2^{2j} \|\Delta_j \delta_{\mathbf{n}}\|_2. \quad (3.12)$$

Denote \mathcal{B} the following form

$$\sum_{i=1,2} \Delta_j (\mathcal{M} \nabla^2 \mathbf{n}^i \delta_{\mathbf{n}}) + \Delta_j (\mathcal{M} \nabla \mathbf{n}^i \nabla \delta_{\mathbf{n}}) + \sum_{i,k=1,2} \Delta_j (\mathcal{M} \nabla \mathbf{n}^i \nabla \mathbf{n}^k \delta_{\mathbf{n}}) + [\Delta_j, \mathcal{M}] \nabla^2 \delta_{\mathbf{n}}.$$

Then by (3.7) and (3.12), we have

$$B_6 \leq C 2^{2j} \|\Delta_j \delta_{\mathbf{n}}\|_2 \|\mathcal{B}\|_2,$$

where $\|\mathcal{B}\|_2$ is bounded by

$$\sum_{i=1,2} \|\Delta_j(\mathcal{M}\nabla^2 \mathbf{n}^i \delta_{\mathbf{n}})\|_2 + \|\Delta_j(\mathcal{M}\nabla \mathbf{n}^i \nabla \delta_{\mathbf{n}})\|_2 + \sum_{i,k=1,2} \|\Delta_j(\mathcal{M}\nabla \mathbf{n}^i \nabla \mathbf{n}^k \delta_{\mathbf{n}})\|_2 + \|[\Delta_j, \mathcal{M}]\nabla^2 \delta_{\mathbf{n}}\|_2.$$

Then it follows from Lemma 3.5–Lemma 3.7 that

$$B_6 \leq \epsilon \sum_{l=j-9}^{j+9} 2^{4l} \|\Delta_l \delta_{\mathbf{n}}\|_2^2 + C 2^{2js} \bar{h}(t) W(t).$$

Moreover,

$$|B_3| + |B_4| + |B_5| \leq C 2^{2j} \|\Delta_j \delta_{\mathbf{n}}\|_2 \|\mathcal{B}\|_2,$$

and

$$|B_3| + |B_4| + |B_5| + |B_6| \leq 4\epsilon \sum_{l=j-9}^{j+9} 2^{4l} \|\Delta_l \delta_{\mathbf{n}}\|_2^2 + C 2^{2js} \bar{h}(t) W(t).$$

Thus, Proposition 3.3 follows from Lemma 3.8 and the estimates for B_i . \square

A Local well-posedness results in \mathbb{R}^d with $d = 2, 3$

The symbol $\langle \cdot, \cdot \rangle$ denotes the integral in \mathbb{R}^d with $d = 2, 3$. Moreover, $\mathcal{P}(\cdot, \dots, \cdot)$ denotes a polynomial depending on the arguments in the parentheses whose order, for example, is less than 10.

In this section, we are aimed to prove the local existence and blow-up criterion for strong solutions of the system (1.3) in \mathbb{R}^d with $d = 2, 3$. Firstly, we use the classical Friedrich's method to construct the approximate solutions of (1.3) as in [38, 36]. The main difficulty lies in the Schrödinger term $\mathbf{n} \times \mathbf{h}$, which can't be controlled by the term $\mathbf{n} \times (\mathbf{n} \times \mathbf{h})$ when $|\mathbf{n}| \neq 1$. Hence, we introduce an equivalent system of (1.3) as follows

$$\partial_t \mathbf{n} = \alpha \mathbf{n} \times (\mathbf{h} \times \mathbf{n}) + \beta \mathbf{n} \times [(\mathbf{n} \times \mathbf{h}) \times \mathbf{n}], \quad (\text{A.1})$$

Secondly, blow-up criterion is similar to [36]. We'll use a better representation formula of $\mathbf{h} \cdot \mathbf{n}$ and the vertical property of $\mathbf{n} \times \mathbf{h}$ with respect to \mathbf{n} .

Our main theorem states as follows.

Theorem A.1 *Let $s \geq 2$ be an integer, and the initial data $\nabla \mathbf{n}_0 \in H^{2s}(\mathbb{R}^d)$ for $d = 2, d = 3$. Then there exist $T > 0$ and a solution \mathbf{n} of the system (1.3) such that*

$$\nabla \mathbf{n} \in C([0, T^*); H^{2s}(\mathbb{R}^d)).$$

Moreover, if T^ is the maximal existence time of the solution, then $T^* < +\infty$ implies that*

$$\int_0^{T^*} \|\nabla \mathbf{n}(t)\|_{L^\infty}^2 dt = +\infty.$$

The following lemma will be frequently used for the commutator; for example see [5].

Lemma A.2 For $\alpha, \beta \in N^3$ or N^2 , it holds that

$$\begin{aligned} \|D^\alpha(fg)\|_{L^2} &\leq C \sum_{|\gamma|=|\alpha|} (\|f\|_{L^\infty} \|D^\gamma g\|_{L^2} + \|g\|_{L^\infty} \|D^\gamma f\|_{L^2}), \\ \|[D^\alpha, f]D^\beta g\|_{L^2} &\leq C \left(\sum_{|\gamma|=|\alpha|+|\beta|} \|D^\gamma f\|_{L^2} \|g\|_{L^\infty} + \sum_{|\gamma|=|\alpha|+|\beta|-1} \|\nabla f\|_{L^\infty} \|D^\gamma g\|_{L^2} \right). \end{aligned}$$

Let a, k_1, k_2, k_3 be the parameters of \mathbf{h} , then we have the following inequality.

Lemma A.3 For any vector $f \in L^2(\mathbb{R}^d)$, there holds

$$(k_2 - a)\|f\|_2^2 + (k_3 - k_2)\|\mathbf{n} \cdot f\|_2^2 \geq 0.$$

In fact, on one hand

- if $a = k_1$, then either $k_3 \geq k_2$ or $|k_3 - k_2| \leq |k_2 - a|$;
- if $a = k_2$, then $k_3 \geq k_2$;
- if $a = k_3$, then $|k_3 - k_2| = |k_2 - a|$,

on the other hand, $|\mathbf{n} \times \mathbf{curl} \mathbf{n}|^2 + |\mathbf{n} \cdot (\mathbf{curl} \mathbf{n})|^2 = |\mathbf{curl} \mathbf{n}|^2$ implies the above inequality.

Proof of Theorem A.1: It's divided into three steps.

Step 1. Construction of the approximated solutions: Let $b \in S^2$ be a constant vector, $\mathbf{n}_0 : \mathbb{R}^d \rightarrow S^2$ such that $\mathbf{n}_0 - b \in H^k(\mathbb{R}^d)$ with $k > 0$. Let

$$\mathcal{J}_\epsilon f = \mathcal{F}^{-1}(\phi(\frac{\xi}{\epsilon})\mathcal{F}f),$$

where $\mathcal{F}(f)(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}dx$ is usual Fourier transform and $\phi(\xi)$ is a smooth cut-off function with $\phi = 1$ in B_1 and $\phi = 0$ outside of B_2 . We construct the approximate system of (A.1),

$$\begin{cases} \partial_t \mathbf{n}_\epsilon = \alpha \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) + \beta \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \\ \mathbf{n}_\epsilon|_{t=0} = \mathcal{J}_\epsilon \mathbf{n}_0, \end{cases} \quad (\text{A.2})$$

where

$$\begin{aligned} \mathcal{J}_\epsilon h_\epsilon &= 2a\Delta \mathcal{J}_\epsilon \mathbf{n}_\epsilon + 2(k_1 - a)\nabla \operatorname{div} \mathcal{J}_\epsilon \mathbf{n}_\epsilon - 2(k_2 - a)\mathbf{curl}(\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathbf{curl} \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) \\ &\quad - 2(k_3 - a)\mathbf{curl}((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \mathbf{curl} \mathcal{J}_\epsilon \mathbf{n}_\epsilon)\mathcal{J}_\epsilon \mathbf{n}_\epsilon) - 2(k_3 - k_2)(\mathbf{curl} \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \mathcal{J}_\epsilon \mathbf{n}_\epsilon) \cdot \mathbf{curl} \mathcal{J}_\epsilon \mathbf{n}_\epsilon. \end{aligned}$$

By the Cauchy-Lipschitz theorem (for example, see [3]), we know that there exists a strictly maximal time T_ϵ and a unique solution $\mathbf{n}_\epsilon - \mathbf{n}_0 \in C([0, T_\epsilon); H^k(\mathbb{R}^d))$ for any $k > 0$.

Step 2. Uniform energy estimates: We consider the evolution of the following energy norm

$$\begin{aligned} E_s(\mathbf{n}_\epsilon) &= \|\mathbf{n}_\epsilon - \mathbf{n}_0\|_2^2 + \int_{\mathbb{R}^d} W(\mathbf{n}_\epsilon, \nabla \mathbf{n}_\epsilon)dx + a\|\Delta^s \nabla \mathbf{n}_\epsilon\|_2^2 + (k_1 - a)\|\Delta^s \operatorname{div} \mathbf{n}_\epsilon\|_2^2 \\ &\quad + (k_2 - a)\|\Delta^s \mathbf{curl} \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_2^2 + (k_3 - a)\|\Delta^s \mathbf{curl} \mathbf{n}_\epsilon \cdot \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_2^2, \end{aligned}$$

and it's sufficient to prove that

$$\frac{d}{dt}E_s(\mathbf{n}_\epsilon) \leq C\mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2\mathbf{n}_\epsilon\|_{L^\infty})E_s(\mathbf{n}_\epsilon) \leq \mathcal{F}(E_s(\mathbf{n}_\epsilon)), \quad (\text{A.3})$$

where we used the embedding equality with $s \geq 2$ and \mathcal{F} is an increasing function with $\mathcal{F}(0) = 0$. Indeed, it means that there exists a $T > 0$ depending only on $E_s(\mathbf{n}_0)$ such that for all $t \in [0, \min(T, T_\epsilon)]$,

$$E_s(\mathbf{n}_\epsilon) \leq 2E_s(\mathbf{n}_0),$$

which implies that $T_\epsilon \geq T$ by a continuous argument. Then the uniform estimates for the solutions \mathbf{n}_ϵ on $[0, T]$ hold which yield that there exists a local solution \mathbf{n} of (A.1) by the standard compactness arguments. Also, if $|\mathbf{n}_0| = 1$, multiply $\cdot \mathbf{n}$ on both sides of (A.1) and we can obtain $|\mathbf{n}| = 1$.

Next, we come to prove the estimate (A.3).

2.1. Lower order terms: In fact, using the equation (A.2) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{n}_\epsilon - \mathbf{n}_0\|_2^2 &= \langle \partial_t \mathbf{n}_\epsilon, \mathbf{n}_\epsilon - \mathbf{n}_0 \rangle \\ &\leq C(1 + \|\mathbf{n}_\epsilon\|_{L^\infty} + \|\nabla \mathbf{n}_\epsilon\|_{L^\infty})^4 (\|\nabla \mathbf{n}_\epsilon\|_2 + \|\Delta \mathbf{n}_\epsilon\|_2) \|\mathbf{n}_\epsilon - \mathbf{n}_0\|_2 \\ &\leq C(1 + \|\mathbf{n}_\epsilon\|_{L^\infty} + \|\nabla \mathbf{n}_\epsilon\|_{L^\infty})^4 E_s(\mathbf{n}_\epsilon) \end{aligned}$$

and on the other hand

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}^d} W(\mathbf{n}_\epsilon, \nabla \mathbf{n}_\epsilon)(\cdot, t) dx \\ &= \int_{\mathbb{R}^d} \left(W_{n^l} - \nabla_i W_{p_i^l} \right) (\mathbf{n}_\epsilon, \nabla \mathbf{n}_\epsilon) \\ &\quad \cdot \mathcal{J}_\epsilon (\alpha (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) + \beta (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon))) dx \\ &\leq C\mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^1}^2, \end{aligned}$$

which are the required estimates.

2.2. Higher order term: Direct calculation shows that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle \nabla \Delta^s \mathbf{n}_\epsilon, \nabla \Delta^s \mathbf{n}_\epsilon \rangle &= -\alpha \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \Delta^{s+1} \mathbf{n}_\epsilon \rangle \\ &\quad - \beta \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \Delta^{s+1} \mathbf{n}_\epsilon \rangle \\ &:= I_1 + I_2, \end{aligned}$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle \text{div} \Delta^s \mathbf{n}_\epsilon, \text{div} \Delta^s \mathbf{n}_\epsilon \rangle &= -\alpha \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \nabla \text{div} \Delta^s \mathbf{n}_\epsilon \rangle \\ &\quad - \beta \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \nabla \text{div} \Delta^s \mathbf{n}_\epsilon \rangle \\ &:= I'_1 + I'_2, \end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \langle \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&= \alpha \langle \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&\quad + \beta \langle \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&\quad + \alpha \langle \mathcal{J}_\epsilon^2 (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&\quad + \beta \langle \mathcal{J}_\epsilon^2 (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&:= I_1'' + I_2'' + I_3'' + I_4'',
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \langle \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&= \alpha \langle \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&\quad + \beta \langle \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathcal{J}_\epsilon (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&\quad + \alpha \langle \mathcal{J}_\epsilon^2 (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&\quad + \beta \langle \mathcal{J}_\epsilon^2 (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)) \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon \rangle \\
&:= I_1''' + I_2''' + I_3''' + I_4'''.
\end{aligned}$$

Then we have

$$\begin{aligned}
& 2(k_2 - a)I_3'' + 2(k_2 - a)I_4'' + 2(k_3 - a)I_3''' + 2(k_3 - a)I_4''' \\
& \leq C\mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2.
\end{aligned}$$

By the formula of $\mathcal{J}_\epsilon \mathbf{n}_\epsilon$ and commutator estimates in Lemma A.2, we get

$$\begin{aligned}
& \|\Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon))\|_{L^2} \\
& \leq \|[\Delta^s, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times] (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)\|_{L^2} + \|\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)\|_{L^2} \\
& \leq \mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) (\|\nabla \mathbf{n}_\epsilon\|_{H^{2s}} + \|\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_2), \tag{A.4}
\end{aligned}$$

Recall the commutator estimates of $[\mathcal{J}_\epsilon, f]$ in [36],

$$\|[\mathcal{J}_\epsilon, f] \nabla g\|_{L^p} \leq C(1 + \|\nabla f\|_{L^\infty}) \|g\|_{L^p},$$

therefore we have

$$\begin{aligned}
& 2aI_1 + 2(k_1 - a)I_1' + 2(k_2 - a)I_1'' + 2(k_3 - a)I_1''' \\
&= \alpha \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), -2a\mathcal{J}_\epsilon \Delta^{s+1} \mathbf{n}_\epsilon - 2(k_1 - a)\mathcal{J}_\epsilon \nabla \operatorname{div} \Delta^s \mathbf{n}_\epsilon \\
&\quad + 2(k_2 - a)\mathcal{J}_\epsilon \mathbf{curl} ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathbf{curl} \mathbf{n}_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon) \\
&\quad + 2(k_3 - a)\mathcal{J}_\epsilon \mathbf{curl} ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \cdot \Delta^s \mathbf{curl} \mathbf{n}_\epsilon) \cdot \mathcal{J}_\epsilon \mathbf{n}_\epsilon) \rangle \\
&\leq -\alpha \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \Delta^s \mathcal{J}_\epsilon h_\epsilon \rangle \\
&\quad + C(\delta) \mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2 + \delta \|\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_{L^2}^2 \\
&\leq -\frac{\alpha}{2} \langle \Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon, \Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon \rangle + C\mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2
\end{aligned}$$

Similarly,

$$\begin{aligned}
& 2aI_2 + 2(k_1 - a)I_2' + 2(k_2 - a)I_2'' + 2(k_3 - a)I_2''' \\
& \leq -\beta \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \Delta^s \mathcal{J}_\epsilon h_\epsilon \rangle \\
& \quad + C(\delta) \mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2 + \delta \|\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_{L^2}^2 \\
& \leq C(\delta) \mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2 + 2\delta \|\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_{L^2}^2,
\end{aligned}$$

where we used the relation

$$\langle (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \Delta^s \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \Delta^s \mathcal{J}_\epsilon h_\epsilon \rangle = 0,$$

and thus

$$\begin{aligned}
& \langle \Delta^s (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times ((\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times \mathcal{J}_\epsilon h_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon)), \Delta^s \mathcal{J}_\epsilon h_\epsilon \rangle \\
& = \langle [\Delta^s, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times)] (\mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon), \Delta^s \mathcal{J}_\epsilon h_\epsilon \rangle \\
& \quad - \langle [\Delta^s, (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times)] \mathcal{J}_\epsilon h_\epsilon, (\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon) \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon \rangle \\
& \leq \langle [\Delta^s, \mathcal{J}_\epsilon \mathbf{n}_\epsilon \times (\mathcal{J}_\epsilon \mathbf{n}_\epsilon \times)] (\nabla \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon), \nabla \Delta^{s-1} \mathcal{J}_\epsilon h_\epsilon \rangle \\
& \quad + C(\delta) \mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2 + \delta \|\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_{L^2}^2 \\
& \leq C(\delta) \mathcal{P}(\|\mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla \mathbf{n}_\epsilon\|_{L^\infty}, \|\nabla^2 \mathbf{n}_\epsilon\|_{L^\infty}) \|\nabla \mathbf{n}_\epsilon\|_{H^{2s}}^2 + 2\delta \|\Delta^s \mathcal{J}_\epsilon h_\epsilon \times \mathcal{J}_\epsilon \mathbf{n}_\epsilon\|_{L^2}^2.
\end{aligned}$$

Combining the above estimates, the inequality (A.3) is satisfied by choosing δ is sufficiently small, thus the proof of the local existence is complete.

Step 3. Blow-up criterion.

Let $T^* < \infty$ be the maximal existence time of the solution. Then it is sufficient to prove that

$$\frac{d}{dt} E_s(\mathbf{n}) \leq C(1 + \|\nabla \mathbf{n}\|_{L^\infty}^2) E_s(\mathbf{n}), \quad (\text{A.5})$$

where

$$\begin{aligned}
E_s(\mathbf{n}) = & \|\mathbf{n} - \mathbf{n}_0\|_{L^2}^2 + \int_{\mathbb{R}^d} W(\mathbf{n}, \nabla \mathbf{n}) dx + a \|\Delta^s \nabla \mathbf{n}\|_{L^2}^2 + (k_1 - a) \|\Delta^s \operatorname{div} \mathbf{n}\|_{L^2}^2 \\
& + (k_2 - a) \|\mathbf{n} \times \Delta^s (\nabla \times \mathbf{n})\|_{L^2}^2 + (k_3 - a) \|\mathbf{n} \cdot \Delta^s (\nabla \times \mathbf{n})\|_{L^2}^2.
\end{aligned}$$

The proof of (A.5) is more subtle with respect to the existence, since we can't use the bound of $\|\nabla^2 \mathbf{n}\|_\infty$. However, at this time we have $|\mathbf{n}| = 1$, and $\mathbf{n} \cdot \Delta \mathbf{n} = -|\nabla \mathbf{n}|^2$.

3.1. Lower order terms: It is easy to see that

$$\begin{aligned}
\frac{d}{dt} \|\mathbf{n} - \mathbf{n}_0\|_2^2 & = \langle \partial_t \mathbf{n}, \mathbf{n} - \mathbf{n}_0 \rangle \\
& = 2 \langle \alpha \mathbf{n} \times (\mathbf{h} \times \mathbf{n}) + \beta \mathbf{n} \times \mathbf{h}, \mathbf{n} - \mathbf{n}_0 \rangle \\
& \leq C(\|\Delta \mathbf{n}\|_2 + \|\nabla \mathbf{n}\|_2) \|\mathbf{n} - \mathbf{n}_0\|_2 \leq C E_s(\mathbf{n}),
\end{aligned}$$

and

$$\frac{d}{dt} \int_{\mathbb{R}^d} W(\mathbf{n}, \nabla \mathbf{n})(\cdot, t) dx = \int_{\mathbb{R}^d} \left(W_{n^l} - \nabla_i W_{p_i^l} \right) \partial_t n^l dx = -\alpha \int_{\mathbb{R}^d} |\mathbf{n} \times \mathbf{h}|^2 dx.$$

Step 3.2. Higher order term: Direct calculation shows that

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \langle \nabla \Delta^s \mathbf{n}, \nabla \Delta^s \mathbf{n} \rangle \\
&= -\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^{s+1} \mathbf{n} \rangle - \beta \langle \Delta^s (\mathbf{n} \times \mathbf{h}), \Delta^{s+1} \mathbf{n} \rangle := I_1 + I_2, \\
& \frac{1}{2} \frac{d}{dt} \langle \operatorname{div} \Delta^s \mathbf{n}, \operatorname{div} \Delta^s \mathbf{n} \rangle \\
&= -\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \nabla \operatorname{div} \Delta^s \mathbf{n} \rangle - \beta \langle \Delta^s (\mathbf{n} \times \mathbf{h}), \nabla \operatorname{div} \Delta^s \mathbf{n} \rangle := I'_1 + I'_2, \\
& \frac{1}{2} \frac{d}{dt} \langle \mathbf{n} \times \Delta^s \operatorname{curl} \mathbf{n}, \mathbf{n} \times \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&= \alpha \langle \mathbf{n} \times \Delta^s \operatorname{curl} (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \mathbf{n} \times \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&\quad + \beta \langle \mathbf{n} \times \Delta^s \operatorname{curl} (\mathbf{n} \times \mathbf{h}), \mathbf{n} \times \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&\quad + \alpha \langle (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) \times \Delta^s \operatorname{curl} \mathbf{n}, \mathbf{n} \times \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&\quad + \beta \langle (\mathbf{n} \times \mathbf{h}) \times \Delta^s \operatorname{curl} \mathbf{n}, \mathbf{n} \times \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&:= I''_1 + I''_2 + I''_3 + I''_4,
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \langle \mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n}, \mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&= \alpha \langle \mathbf{n} \cdot \Delta^s \operatorname{curl} (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&\quad + \beta \langle \mathbf{n} \cdot \Delta^s \operatorname{curl} (\mathbf{n} \times \mathbf{h}), \mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&\quad + \alpha \langle (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})) \times \Delta^s \operatorname{curl} \mathbf{n}, \mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&\quad + \beta \langle (\mathbf{n} \times \mathbf{h}) \times \Delta^s \operatorname{curl} \mathbf{n}, \mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n} \rangle \\
&:= I'''_1 + I'''_2 + I'''_3 + I'''_4.
\end{aligned}$$

For the terms I_1, I'_1, I''_1, I'''_1 , we have

$$\begin{aligned}
& 2aI_1 + 2(k_1 - a)I'_1 + 2(k_2 - a)I''_1 + 2(k_3 - a)I'''_1 \\
&= \alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), -2a\Delta^{s+1} \mathbf{n} - 2(k_1 - a)\nabla \operatorname{div} \Delta^s \mathbf{n} + 2(k_2 - a)\operatorname{curl} \operatorname{curl} \Delta^s \mathbf{n} \\
&\quad + \alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), 2(k_3 - k_2)[(\mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n})\operatorname{curl} \mathbf{n} + \nabla(\mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n}) \times \mathbf{n}] \rangle \\
&= -\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s \nabla_\alpha W_{p_\alpha}^k \rangle \\
&\quad + 2(k_3 - k_2)\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), (\mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n})\operatorname{curl} \mathbf{n} - \Delta^s ((\mathbf{n} \cdot \operatorname{curl} \mathbf{n})\operatorname{curl} \mathbf{n}) \rangle \\
&\quad + 2(k_3 - k_2)\alpha \langle \Delta^s (\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \nabla(\mathbf{n} \cdot \Delta^s \operatorname{curl} \mathbf{n}) \times \mathbf{n} - \nabla \Delta^s (\mathbf{n} \cdot \operatorname{curl} \mathbf{n}) \times \mathbf{n} \rangle \\
&\quad + 2(k_3 - k_2)\alpha \langle \nabla \Delta^s (\mathbf{n} \cdot \operatorname{curl} \mathbf{n}) \times \mathbf{n} - \Delta^s (\nabla(\mathbf{n} \cdot \operatorname{curl} \mathbf{n}) \times \mathbf{n}) \rangle, \tag{A.6}
\end{aligned}$$

where we have used the following relation, for a function f and a vector field u , there holds

$$\operatorname{curl}(fu) = f\operatorname{curl}u + \nabla f \times u.$$

We will use the following Gagliardo-Sobolev inequality on \mathbb{R}^d (for example, see [1]). Let $\tau \in N$, and $\tau \geq 2s - 1$, then for $1 \leq j \leq [\tau/2]$, $[\tau/2] + 1 \leq k \leq \tau$, and $f \in H^{\tau+1}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\|\nabla^j f\|_{L^\infty} &\leq C \|\nabla f\|_{H^\tau}^{\frac{j}{\tau+1-d/2}} \|f\|_{L^\infty}^{1-\frac{j}{\tau+1-d/2}}, \\
\|\nabla^k f\|_{L^2} &\leq C \|\nabla f\|_{H^\tau}^{\frac{k-d/2}{\tau+1-d/2}} \|f\|_{L^\infty}^{1-\frac{k-d/2}{\tau+1-d/2}}.
\end{aligned}$$

Hence, for $\tau \geq 2s - 1$ with $s \geq 2$, the following inequalities hold,

$$\begin{aligned} \|\nabla^{\tau+1}\mathbf{n}\|_{L^2}\|\nabla\mathbf{n}\|_{L^\infty} + \|\nabla^\tau\mathbf{n}\|_{L^2}\|\nabla^2\mathbf{n}\|_{L^\infty} + \|\nabla^\tau\mathbf{n}\|_{L^2}\|\nabla\mathbf{n}\|_{L^\infty}^2 &\leq C\|\nabla\mathbf{n}\|_{H^{\tau+1}}, \\ (\|\nabla^2\mathbf{n}\|_{L^\infty} + \|\nabla\mathbf{n}\|_{L^\infty}^2)\|\nabla^\tau\mathbf{n}\|_{L^2} &\leq C\|\nabla\mathbf{n}\|_{L^\infty}\|\nabla^{\tau+1}\mathbf{n}\|_{L^2}. \end{aligned} \quad (\text{A.7})$$

By Lemma A.2 and Gagliardo-Sobolev inequality (A.7), we have

$$\begin{aligned} &2aI_1 + 2(k_1 - a)I_1' + 2(k_2 - a)I_1'' + 2(k_3 - a)I_1''' \\ &\leq -\alpha\langle\Delta^s(\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s\nabla_\alpha W_{p_\alpha^l}\rangle \\ &\quad + C(\|\Delta^s\mathbf{n}\|_{L^2} + \|\Delta^s\mathbf{n}\|_{L^2}(\|\nabla\mathbf{n}\|_{L^\infty}^2 + \|\nabla^2\mathbf{n}\|_{L^\infty}) + \|\Delta^s\mathbf{h} \times \mathbf{n}\|_{L^2}) \\ &\quad \cdot (\|\Delta^s\mathbf{n}\|_{L^2}\|\nabla\mathbf{n}\|_{L^\infty}^2 + \|\nabla\mathbf{n}\|_{L^\infty}\|\Delta^s\nabla\mathbf{n}\|_{L^2} + \|\Delta^s\mathbf{n}\|_{L^2}\|\nabla^2\mathbf{n}\|_{L^\infty}) \\ &\leq -\alpha\langle\Delta^s(\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s\nabla_\alpha W_{p_\alpha^l}\rangle + C_\delta\|\nabla\mathbf{n}\|_{L^\infty}^2\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2. \end{aligned} \quad (\text{A.8})$$

Note that

$$\begin{aligned} &-\alpha\langle\Delta^s(\mathbf{n} \times (\mathbf{h} \times \mathbf{n})), \Delta^s\nabla_\alpha W_{p_\alpha^l}\rangle \\ &= -\alpha\langle\Delta^s\left(\mathbf{n} \times \left((\mathbf{h} - \nabla_\alpha W_{p_\alpha^k}) \times \mathbf{n}\right)\right), \Delta^s\nabla_\alpha W_{p_\alpha^l}\rangle \\ &\quad -\alpha\langle\Delta^s\left(\mathbf{n} \times (\nabla_\alpha W_{p_\alpha^k} \times \mathbf{n})\right), \Delta^s(2a\Delta\mathbf{n})\rangle \\ &\quad -\alpha\langle\Delta^s\left(\mathbf{n} \times ((\nabla_\alpha W_{p_\alpha^k} - 2a\Delta\mathbf{n}) \times \mathbf{n})\right), \Delta^s(\nabla_\alpha W_{p_\alpha^l} - 2a\Delta\mathbf{n})\rangle \\ &\quad -2a\alpha\langle\Delta^s(\mathbf{n} \times (\Delta\mathbf{n} \times \mathbf{n})), \Delta^s(\nabla_\alpha W_{p_\alpha^l} - 2a\Delta\mathbf{n})\rangle \\ &\doteq I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned} \quad (\text{A.9})$$

Note that $\mathbf{h} - \nabla_\alpha W_{p_\alpha^k} = -W_{\mathbf{n}_l} = -2(k_3 - k_2)(\mathbf{n} \cdot \mathbf{curl}\mathbf{n})\mathbf{curl}\mathbf{n}$, we have

$$I_{11} \leq C_\delta(1 + \|\nabla\mathbf{n}\|_{L^\infty}^2)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2,$$

and similar estimates hold for the term I_{13} , since I_{13} can be written as the sum of a nonnegative term and a commutator term. As to I_{14} , by Lemma (A.2) and (A.3) we have

$$\begin{aligned} I_{14} &\leq -4a(k_1 - a)\alpha\langle\nabla\Delta^s\text{div}\mathbf{n}, \nabla\Delta^s\text{div}\mathbf{n}\rangle - 4a(k_2 - a)\alpha\langle\nabla\Delta^s\mathbf{curl}\mathbf{n}, \nabla\Delta^s\mathbf{curl}\mathbf{n}\rangle \\ &\quad - 4a(k_3 - k_2)\alpha\langle\mathbf{n} \cdot \nabla_l\Delta^s\mathbf{curl}\mathbf{n}, \mathbf{n} \cdot \nabla_l\Delta^s\mathbf{curl}\mathbf{n}\rangle \\ &\quad + C_\delta(\|\nabla\mathbf{n}\|_{L^\infty}^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2 \\ &\leq C_\delta(\|\nabla\mathbf{n}\|_{L^\infty}^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2. \end{aligned}$$

At last, we estimate I_{12} . Direct calculation shows that

$$\begin{aligned} &\nabla_\alpha W_{p_\alpha^l} \cdot n^l \\ &= -2k_2|\nabla\mathbf{n}|^2 - 2(k_3 - k_2)(\mathbf{n} \cdot \mathbf{curl}\mathbf{n})^2 - 2(k_1 - k_2)(\text{div}\mathbf{n})^2 + 2(k_1 - k_2)\nabla_l(n^l\text{div}\mathbf{n}). \end{aligned} \quad (\text{A.10})$$

Thus, by Lemma A.2 and (A.7) we infer that

$$\begin{aligned}
I_{12} &= -2a\alpha\langle\Delta^s\nabla_\alpha W_{p_\alpha^l}, \Delta^{s+1}\mathbf{n}\rangle + 2a\alpha\langle\Delta^s((\nabla_\alpha W_{p_\alpha^l} \cdot n^l)\mathbf{n}), \Delta^{s+1}\mathbf{n}\rangle \\
&= -4a^2\alpha\langle\Delta^{s+1}\mathbf{n}, \Delta^{s+1}\mathbf{n}\rangle - 4a(k_1 - a)\alpha\langle\nabla\Delta^s\operatorname{div}\mathbf{n}, \nabla\Delta^s\operatorname{div}\mathbf{n}\rangle \\
&\quad - 4a(k_2 - a)\alpha\langle\nabla\Delta^s(\nabla \times \mathbf{n}), \nabla\Delta^s(\nabla \times \mathbf{n})\rangle \\
&\quad - 4a(k_3 - k_2)\alpha\langle\nabla_l\Delta^s(\mathbf{n} \cdot (\nabla \times \mathbf{n})), \mathbf{n} \cdot \nabla_l\Delta^s(\nabla \times \mathbf{n})\rangle \\
&\quad - 4a(k_3 - k_2)\alpha\langle[\nabla_l\Delta^s, \mathbf{n}](\mathbf{n} \cdot \nabla \times \mathbf{n}), \nabla_l\Delta^s(\nabla \times \mathbf{n})\rangle \\
&\quad + 2a\alpha\langle\Delta^s((\nabla_\alpha W_{p_\alpha^l} \cdot n^l)\mathbf{n}), \Delta^{s+1}\mathbf{n}\rangle \\
&\leq -4a^2\alpha\langle\Delta^{s+1}\mathbf{n}, \Delta^{s+1}\mathbf{n}\rangle + C_\delta(\|\nabla\mathbf{n}\|_{L^\infty}^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_{L^2}^2 + 2\delta\|\Delta^{s+1}\mathbf{n}\|_{L^2}^2,
\end{aligned}$$

where we used Lemma (A.2)-(A.3) and for the last term of the second equality, we have the following observation with the help of (A.10):

$$\begin{aligned}
&\langle\Delta^s(\nabla_l(n^l\operatorname{div}\mathbf{n}) \cdot n^k), \Delta^{s+1}n^k\rangle \\
&= \langle\Delta^s\nabla_l(n^l\operatorname{div}\mathbf{n}), \Delta^s(n^k\Delta n^k)\rangle - \langle\Delta^s\nabla_l(n^l\operatorname{div}\mathbf{n}), [\Delta^s, n^k]\Delta n^k\rangle \\
&\quad + \langle[\Delta^s, n^k]\nabla_l(n^l\operatorname{div}\mathbf{n}), \Delta^{s+1}n^k\rangle.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&2aI_2 + 2(k_1 - a)I_2' + 2(k_2 - a)I_2'' + 2(k_3 - a)I_2''' \\
&\leq C_\delta(\|\nabla\mathbf{n}\|_\infty^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_2^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_2^2,
\end{aligned}$$

and

$$|I_3''| + |I_4''| + |I_3'''| + |I_4'''| \leq C_\delta(\|\nabla\mathbf{n}\|_\infty^2 + 1)\|\Delta^s\nabla\mathbf{n}\|_2^2 + \delta\|\Delta^{s+1}\mathbf{n}\|_2^2.$$

Thus, the above arguments show that (A.5) is true. \square

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